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Multidimensional Sinusoidal Frequency Estimation Using Subspace and Projection Separation Approaches

Longting Huang, Yuntao Wu, H. C. So, Yanduo Zhang, and Lei Huang

***Abstract*—In this correspondence, a computationally efficient method that combines the subspace and projection separation approaches is developed for R-dimensional (R-D) frequency estimation of multiple sinusoids, where $R \geq 3$, in the presence of white Gaussian noise. Through extracting a 2-D slice matrix set from the multidimensional data, we devise a covariance matrix associated with one dimension, from which the corresponding frequencies are estimated using the root-MUSIC method. With the use of the frequency estimates in this dimension, a set of projection separation matrices is then constructed to separate all frequencies in the remaining dimensions. Root-MUSIC method is again applied to estimate these single-tone frequencies while multidimensional frequency pairing is automatically attained. Moreover, the mean square error of the frequency estimator is derived and confirmed by computer simulations. It is shown that the proposed approach is superior to two state-of-the-art frequency estimators in terms of accuracy and computational complexity.**

***Index Terms*—Multidimensional frequency estimation, subspace method, projection separation.**

I. Introduction

The topic of R-dimensional (R-D) frequency estimation, where $R \geq 3$, has received extensive attention for its widespread applications in numerous fields such as MIMO wireless channel sounding [1], mobile communications [2], MIMO radar [3], sonar, seismology and nuclear magnetic resonance spectroscopy [4]. Many high-resolution subspace-based parameter estimation techniques have been proposed to solve this problem such as multidimensional folding (MDF) [1], unitary ESPRIT [5], improved MDF (IMDF) [6], MUSIC [7], rank reduction estimator (RARE) [8], decoupled root-MUSIC [9] and higher-order singular value decomposition (HOSVD) [10] methods. Unitary ESPRIT, MDF and IMDF techniques are based on the conventional ESPRIT approach where it is difficult to directly utilize the original multidimensional data. Typically, they require to enlarge the received data to construct a 2-D matrix with larger size, and then employ the ESPRIT-based method to obtain the desired frequency pairs. Consequently, their computational burden is very heavy particularly when the data size is large. The RARE algorithm vectorizes the observed data to exploit the Vandermonde structure and matrix polynomials are constructed to find the frequencies of each dimension. However, computing the polynomial coefficients is a highly demanding job. In the decoupled root-MUSIC

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L. Huang and H. C. So are with the Department of Electronic Engineering, City University of Hong Kong, Hong Kong 852, China.

Y. Wu and Y. Zhang are with the School of Computer Science and Engineering, Wuhan Institute of Technology, Wuhan 430073, China (e-mail: ytwu@sina.com).

L. Huang is with the Shenzhen Graduate School, Harbin Institute of Technology, Shenzhen 518055, China.

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algorithm, R-D harmonic retrieval is decomposed into R 1-D problems by tensor decomposition [11], [12], which significantly reduces the computational load, but pairing of the R-D frequencies is required. On the other hand, the HOSVD method utilizes the structure inherent in the received data at the expense of a high computational complexity. In this work, our main contribution is to devise an accurate and computationally efficient estimator for multidimensional frequencies in the presence of white Gaussian noise with the use of the subspace and projection separation techniques.

The rest of this correspondence is organized as follows. The development of the proposed estimator is presented in Section II. There are two basic estimation steps as follows. We first extract a 2-D slice matrix set from the R-D signal to construct a covariance matrix associated with the first dimension, from which the corresponding frequencies are estimated using the root-MUSIC method. With the use of the frequency estimates in the first dimension, a set of projection separation matrices is then devised to separate all frequencies in the remaining dimensions. Root-MUSIC method is again utilized to find these single-tone frequencies while multidimensional frequency pairing is automatically attained. In Section III, the computational complexity and mean square error (MSE) of the devised estimator are analyzed. Since our method exploits covariance matrices whose size is characterized by the number of sinusoids, its computational requirement is small when compared with the conventional schemes. Section IV includes numerical examples for validating the theoretical findings and evaluating the proposed approach by comparing with the IMDF [6] and HOSVD [10] methods as well as Cramér-Rao lower bound (CRLB). Finally, conclusions are drawn in Section V.

II. Proposed Method

A. Signal Model

The observed R-D signal model is

$$x_{m_1,m_2,\ldots,m_R} = s_{m_1,m_2,\ldots,m_R} + q_{m_1,m_2,\ldots,m_R},\quad m_r = 1,2,\ldots,M_r,\ r = 1,2,\ldots,R \tag{1}$$

where

$$s_{m_1,m_2,\ldots,m_R} = \sum_{f=1}^{F} \alpha_f \prod_{r=1}^{R} e^{j\omega_{f,r} m_r} \tag{2}$$

with F being the number of sinusoids, which is assumed to be known *a priori* [13]. Here, α_f is the unknown complex amplitude of fth tone, $\omega_{f,r} \in (-\pi,\pi)$ is the unknown frequency of fth component in the rth dimension. The $q_{m_1,m_2,\ldots,m_R}$ is a R-D circular complex white Gaussian noise with mean zero and unknown variance σ_q^2. It is assumed that $M_r \geq F$, $r = 1,2,\ldots,R$, and the frequencies are distinct in at least one of the R dimensions. Given the $M = \prod_{r=1}^{R} M_r$ samples of $x_{m_1,m_2,\ldots,m_R}$, our task is to find the RF unknown frequency parameters, namely, $\{\omega_{f,r}\}$, $f = 1,2,\ldots,F$, $r = 1,2,\ldots,R$.

B. Estimation in First Dimension

For ease of presentation but without loss of generality, we start frequency estimation in the first dimension with the assumption that $\omega_{k,1} \neq \omega_{l,1}$, $k \neq l$. In tensor form, (1) is

$$\boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{S}} + \boldsymbol{\mathcal{Q}} \tag{3}$$

where $\{\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{S}},\boldsymbol{\mathcal{Q}}\} \in \mathbb{C}^{M_1\times M_2\times\cdots\times M_R}$, $[\boldsymbol{\mathcal{X}}]_{m_1,m_2,\ldots,m_R} = x_{m_1,m_2,\ldots,m_R}$, $[\boldsymbol{\mathcal{S}}]_{m_1,m_2,\ldots,m_R} = s_{m_1,m_2,\ldots,m_R}$ and $[\boldsymbol{\mathcal{Q}}]_{m_1,m_2,\ldots,m_R} = q_{m_1,m_2,\ldots,m_R}$. That is, $\boldsymbol{\mathcal{S}}$ and $\boldsymbol{\mathcal{Q}}$ are the noise-free and noise-only components of $\boldsymbol{\mathcal{X}}$, respectively.

To reduce the dimension of $\boldsymbol{\mathcal{X}}$, we define $\boldsymbol{\mathcal{X}}_{r_1,r_2}$, which is a set of 2-D slice matrices extracting from $\boldsymbol{\mathcal{X}}$, as follows:

$$\boldsymbol{\mathcal{X}}_{r_1,r_2} = \{\boldsymbol{\mathcal{X}}(m_1,\ldots,m_{r_1-1},:,m_{r_1+1},\ldots,m_{r_2-1},:,m_{r_2+1},\ldots,m_R)\} \tag{4}$$

where

$$m_r = 1,2,\ldots,M_r,\quad r = 1,2,\ldots,R \text{ and } r \neq r_1,r_2$$
$$[\boldsymbol{\mathcal{X}}(m_1,\ldots,m_{r_1-1},:,m_{r_1+1},\ldots,m_{r_2-1},:,m_{r_2+1},\ldots,m_R)]_{m_{r_1},m_{r_2}} = x_{m_1,m_2,\ldots,m_R}.$$

Similarly, $\boldsymbol{\mathcal{S}}_{1,r}$ is a noise-free 2-D matrix set extracting from $\boldsymbol{\mathcal{S}}$ and each of the $\boldsymbol{\mathcal{S}}(:,\ldots,m_{r-1},:,m_{r+1},\ldots,m_R) \in \mathbb{C}^{M_1\times M_r}$ has the form of (5), shown at the bottom of the page.

It is easy to verify that $\boldsymbol{\mathcal{S}}(:,\ldots,m_{r-1},:,m_{r+1},\ldots,m_R)$ can be written as

$$\boldsymbol{\mathcal{S}}(:,\ldots,m_{r-1},:,m_{r+1},\ldots,m_R) = \mathbf{G}_1\,\boldsymbol{\Gamma}_{1,r}(m_2,\ldots,m_{r-1},m_{r+1},\ldots,m_R)\,\mathbf{G}_r^H \tag{6}$$

where

$$\mathbf{G}_r = [\mathbf{g}_{1,r},\mathbf{g}_{2,r},\ldots,\mathbf{g}_{F,r}],\ r = 1,2,\ldots,R \tag{7}$$

$$\mathbf{g}_{f,r} = \left[a_{f,r},a_{f,r}^2,\ldots a_{f,r}^{M_r}\right]^T \tag{8}$$

$$a_{f,1} = e^{j\omega_{f,1}},\ a_{f,r} = e^{-j\omega_{f,r}},\ r = 2,3,\ldots,R \tag{9}$$

$$\boldsymbol{\Gamma}_{1,r}(m_2,\ldots,m_{r-1},m_{r+1},\ldots,m_R) = \mathrm{diag}\left(\alpha_1\prod_{\substack{n=2\\n\neq r}}^{R} a_{1,n}^{-m_n},\alpha_2\prod_{\substack{n=2\\n\neq r}}^{R} a_{2,n}^{-m_n},\ldots,\alpha_F\prod_{\substack{n=2\\n\neq r}}^{R} a_{F,n}^{-m_n}\right) \tag{10}$$

$$\begin{bmatrix}
\sum_{f=1}^{F}\alpha_f\prod_{\substack{n=2\\n\neq r}}^{R} e^{j\omega_{f,n}m_n}\cdot e^{j(\omega_{f,1}+\omega_{f,r})} & \cdots & \sum_{f=1}^{F}\alpha_f\prod_{\substack{n=2\\n\neq r}}^{R} e^{j\omega_{f,n}m_n}\cdot e^{j(\omega_{f,1}+M_r\omega_{f,r})} \\
\sum_{f=1}^{F}\alpha_f\prod_{\substack{n=2\\n\neq r}}^{R} e^{j\omega_{f,n}m_n}\cdot e^{j(2\omega_{f,1}+\omega_{f,r})} & \cdots & \sum_{f=1}^{F}\alpha_f\prod_{\substack{n=2\\n\neq r}}^{R} e^{j\omega_{f,n}m_n}\cdot e^{j(2\omega_{f,1}+M_r\omega_{f,r})} \\
\vdots & \ddots & \vdots \\
\sum_{f=1}^{F}\alpha_f\prod_{\substack{n=2\\n\neq r}}^{R} e^{j\omega_{f,n}m_n}\cdot e^{j(M_1\omega_{f,1}+\omega_{f,r})} & \cdots & \sum_{f=1}^{F}\alpha_f\prod_{\substack{n=2\\n\neq r}}^{R} e^{j\omega_{f,n}m_n}\cdot e^{j(M_1\omega_{f,1}+M_r\omega_{f,r})}
\end{bmatrix}. \tag{5}$$

with $(\cdot)^H$, $(\cdot)^T$ and $\mathrm{diag}(\mathbf{a})$ being the conjugate transpose, transpose, and diagonal matrix with vector $\mathbf{a}$ as its main diagonal, respectively.

Define $\hat{\mathbf{R}}_1$ as the sample covariance matrix of matrix set $\boldsymbol{\mathcal{X}}_{1,2}$, which is computed as

$$\hat{\mathbf{R}}_1 = \frac{M_1 M_2}{M} \sum_{m_3=1}^{M_3} \sum_{m_4=1}^{M_4} \cdots \sum_{m_R=1}^{M_R} \boldsymbol{\mathcal{X}}(:,:,m_3,\ldots,m_R) \cdot \boldsymbol{\mathcal{X}}^H(:,:,m_3,\ldots,m_R). \quad (11)$$

According to the structure of slice matrix $\boldsymbol{\mathcal{X}}(:,:,m_3,\ldots,m_R)$, the expected value of $\hat{\mathbf{R}}_1$, denoted by $\mathbf{R}_1$, is

$$\begin{aligned}\mathbf{R}_1 &= \frac{M_1 M_2}{M} \sum_{m_3=1}^{M_3} \cdots \sum_{m_R=1}^{M_R} \mathbf{G}_1 \boldsymbol{\Gamma}_{1,2}(m_3,\ldots,m_R)\mathbf{G}_2^H \\ &\quad \cdot \mathbf{G}_2 \boldsymbol{\Gamma}_{1,2}^H(m_3,\ldots,m_R)\mathbf{G}_1^H + \sigma_q^2 \mathbf{I}_{M_1} \\ &= \frac{M_1 M_2}{M} \mathbf{G}_1 \left(\sum_{m_3=1}^{M_3} \cdots \sum_{m_R=1}^{M_R} \boldsymbol{\Gamma}_{1,2}(m_3,\ldots,m_R)\mathbf{G}_2^H \right. \\ &\quad \left. \cdot \mathbf{G}_2 \boldsymbol{\Gamma}_{1,2}^H(m_3,\ldots,m_R) \right) \mathbf{G}_1^H + \sigma_q^2 \mathbf{I}_{M_1} \\ &= \mathbf{G}_1 \mathbf{B}_1 \mathbf{G}_1^H + \sigma_q^2 \mathbf{I}_{M_1} \end{aligned} \quad (12)$$

where

$$\mathbf{B}_1 = \frac{M_1 M_2}{M} \sum_{m_3=1}^{M_3} \cdots \sum_{m_R=1}^{M_R} \boldsymbol{\Gamma}_{1,2}(m_3,\ldots,m_R)\mathbf{G}_2^H \cdot \mathbf{G}_2 \boldsymbol{\Gamma}_{1,2}^H(m_3,\ldots,m_R) \in \mathbb{C}^{F\times F} \quad (13)$$

and $\mathbf{I}_{M_1}$ is the $M_1 \times M_1$ identity matrix.

In order to circumvent the problem of degraded estimation performance in case of closely spaced frequencies in the same dimension, we further propose the forward-backward (FB) smoothing [14], [15] covariance matrix of $\hat{\mathbf{R}}_1$:

$$\hat{\mathbf{R}}_1^{\mathrm{FB}} = \frac{1}{2}\left(\hat{\mathbf{R}}_1 + \mathbf{J}_{M_1} \hat{\mathbf{R}}_1^* \mathbf{J}_{M_1}\right) \quad (14)$$

where $\mathbf{J}_{M_1} \in \mathbb{C}^{M_1 \times M_1}$ is the exchange matrix with ones on its antidiagnoal and zeros elsewhere.

The expected value of $\hat{\mathbf{R}}_1^{\mathrm{FB}}$ is

$$\mathbf{R}_1^{\mathrm{FB}} = \frac{1}{2}\left(\mathbf{R}_1 + \mathbf{J}_{M_1} \mathbf{R}_1^* \mathbf{J}_{M_1}\right) = \mathbf{G}_1 \tilde{\mathbf{B}}_1 \mathbf{G}_1^H + \sigma_q^2 \mathbf{I}_{M_1} \quad (15)$$

where

$$\tilde{\mathbf{B}}_1 = \frac{1}{2}\left(\mathbf{B}_1 + \mathbf{D}_1 \mathbf{B}_1^* \mathbf{D}_1\right) \quad (16)$$

$$\mathbf{D}_1 = \mathrm{diag}\left(e^{-jM_1\omega_{1,1}}, e^{-jM_1\omega_{2,1}}, \ldots, e^{-jM_1\omega_{F,1}}\right). \quad (17)$$

On the other hand, $\hat{\mathbf{R}}_1^{\mathrm{FB}}$ can be decomposed using eigenvalue decomposition (EVD) as

$$\hat{\mathbf{R}}_1^{\mathrm{FB}} = \hat{\mathbf{V}}_{1s} \hat{\boldsymbol{\Lambda}}_{1s} \hat{\mathbf{V}}_{1s}^H + \hat{\mathbf{V}}_{1n} \hat{\boldsymbol{\Lambda}}_{1n} \hat{\mathbf{V}}_{1n}^H \quad (18)$$

where the column vectors of $\hat{\mathbf{V}}_{1s} \in \mathbb{C}^{M_1 \times F}$ and $\hat{\mathbf{V}}_{1n} \in \mathbb{C}^{M_1 \times (M_1 - F)}$ are the eigenvectors that span the signal and noise subspaces of $\hat{\mathbf{R}}_1$, respectively, with the associated eigenvalues being the diagonal elements of $\hat{\boldsymbol{\Lambda}}_{1s}$ and $\hat{\boldsymbol{\Lambda}}_{1n}$.

Let

$$\hat{\mathbf{E}}_1 = \mathbf{I}_{M_1} - \hat{\mathbf{V}}_{1s} \hat{\mathbf{V}}_{1s}^H. \quad (19)$$

Employing root-MUSIC method to estimate the frequencies based on $\hat{\mathbf{E}}_1$, a null-spectrum function is constructed as:

$$f_1(z) = \beta_1^T(z^{-1})\, \hat{\mathbf{E}}_1 \beta_1(z) \quad (20)$$

where $\beta_1(z) = [z, z^2, \ldots, z^{M_1}]^T$. The polynomial $f_1(z)$ has $2(M_1 - 1)$ roots and the first dimension frequency estimates $\{\hat{\omega}_{f,1}\}$ are obtained from the F largest-magnitude roots inside the unit circle according to $z = e^{j\omega}$.

C. Estimation in Remaining Dimensions

With the use of $\{\hat{\omega}_{f,1}\}$, a set of projection matrices $\hat{\mathbf{P}}_f^{\perp}, f = 1,2,\ldots,F$, is constructed to estimate all F frequencies one by one in each of the rth dimension, $r = 2,3,\ldots,R$. The $\hat{\mathbf{P}}_f^{\perp}$ is defined as

$$\hat{\mathbf{P}}_f^{\perp} = \mathbf{I}_{M_1} - \mathbf{H}_f \left(\mathbf{H}_f^H \mathbf{H}_f\right)^{-1} \mathbf{H}_f^H, \quad f = 1,2,\ldots,F \quad (21)$$

where

$$\mathbf{H}_f = [\mathbf{h}(\hat{\omega}_{1,1}),\ldots,\mathbf{h}(\hat{\omega}_{f-1,1}),\mathbf{h}(\hat{\omega}_{f+1,1}),\ldots,\mathbf{h}(\hat{\omega}_{F,1})] \quad (22)$$

$$\mathbf{h}(\hat{\omega}_{k,1}) = \left[\hat{z}_{k,1}, \hat{z}_{k,1}^2, \ldots, \hat{z}_{k,1}^{M_1}\right]^T, \quad \hat{z}_{k,1} = e^{j\hat{\omega}_{k,1}}. \quad (23)$$

Analogous to (4), we construct the 2-D slice matrices set $\boldsymbol{\mathcal{X}}_{1,r}$ and then use the projection matrix $\hat{\mathbf{P}}_f^{\perp}$ on $\boldsymbol{\mathcal{X}}(:,\ldots,m_{r-1},:,m_{r+1},\ldots,m_R) \in \mathbb{C}^{M_1 \times M_r}$ to obtain $\boldsymbol{\mathcal{X}}_{f,r}(:,\ldots,m_{r-1},:,m_{r+1},\ldots,m_R)$ which contains only the information of $\{\omega_{f,r}\}$ in the rth dimension associated with $\hat{\omega}_{f,1}$.

The $\boldsymbol{\mathcal{X}}_{f,r}(:,\ldots,m_{r-1},:,m_{r+1},\ldots,m_R)$ is defined as

$$\begin{aligned}&\boldsymbol{\mathcal{X}}_{f,r}(:,\ldots,m_{r-1},:,m_{r+1},\ldots,m_R) \\ &\quad = \hat{\mathbf{P}}_f^{\perp} \boldsymbol{\mathcal{X}}(:,\ldots,m_{r-1},:,m_{r+1},\ldots,m_R) \\ &\quad = \hat{\mathbf{P}}_f^{\perp} \mathbf{G}_1 \boldsymbol{\Gamma}_{1,r}(m_2,\ldots,m_{r-1},m_{r+1},\ldots,m_R)\, \mathbf{G}_r^H \\ &\qquad + \hat{\mathbf{P}}_f^{\perp} \boldsymbol{\mathcal{Q}}(:,\ldots,m_{r-1},:,m_{r+1},\ldots,m_R) \end{aligned} \quad (24)$$

where $r = 2,3,\ldots,R,\ f = 1,2,\ldots,F$.

Similar to (11), we define a covariance matrix $\hat{\mathbf{R}}_{f,r}$

$$\begin{aligned}\hat{\mathbf{R}}_{f,r} &= \frac{M_1 M_r}{M} \sum_{m_2=1}^{M_2} \cdots \sum_{m_{r-1}=1}^{M_{r-1}} \sum_{m_{r+1}=1}^{M_{r+1}} \cdots \\ &\quad \sum_{m_R=1}^{M_R} \boldsymbol{\mathcal{X}}_{f,r}^H(:,\ldots,m_{r-1},:,m_{r+1},\ldots,m_R) \\ &\quad \cdot \boldsymbol{\mathcal{X}}_{f,r}(:,\ldots,m_{r-1},:,m_{r+1},\ldots,m_R) \end{aligned} \quad (25)$$

where $r = 2,3,\ldots,R,\ f = 1,2,\ldots,F$.

The expected value of $\hat{\mathbf{R}}_{f,r}$ is calculated as

$$\begin{aligned}\mathbf{R}_{f,r} &= \frac{M_1 M_r}{M} \sum_{m_2=1}^{M_2} \cdots \sum_{m_{r-1}=1}^{M_{r-1}} \sum_{m_{r+1}=1}^{M_{r+1}} \cdots \sum_{m_R=1}^{M_R} \mathbf{G}_r \\ &\quad \times \boldsymbol{\Gamma}_{1,r}^H(m_2,\ldots,m_{r-1},m_{r+1},\ldots,m_R)\mathbf{G}_1^H {\mathbf{P}_f^{\perp}}^H \mathbf{P}_f^{\perp} \\ &\quad \cdot \mathbf{G}_1 \boldsymbol{\Gamma}_{1,r}(m_2,\ldots,m_{r-1},m_{r+1},\ldots,m_R)\mathbf{G}_r^H + \sigma_q^2 \mathbf{I}_{M_r} \\ &= \mathbf{G}_r \mathbf{B}_{f,r} \mathbf{G}_r^H + \sigma_q^2 \mathbf{I}_{M_r} \end{aligned} \quad (26)$$

where

$$\mathbf{B}_{f,r} = \frac{M_1 M_r}{M} \sum_{m_2=1}^{M_2} \cdots \sum_{m_{r-1}=1}^{M_{r-1}} \sum_{m_{r+1}=1}^{M_{r+1}} \cdots \sum_{m_R=1}^{M_R} \mathbf{\Gamma}_{1,r}^H(m_2,\ldots,m_{r-1},m_{r+1},\ldots,m_R) \times \mathbf{G}_1^H \mathbf{P}_f^{\perp H} \cdot \mathbf{P}_f^{\perp} \mathbf{G}_1 \mathbf{\Gamma}_{1,r}(m_2,\ldots,m_{r-1},m_{r+1},\ldots,m_R) \in \mathbb{C}^{F\times F}$$
$$r = 2,3,\ldots,R, \quad f = 1,2,\ldots,F \tag{27}$$

and

$$\mathbf{P}_f^{\perp}\mathbf{G}_1 = [\mathbf{0},\mathbf{0},\ldots,\mathbf{P}_f^{\perp}\mathbf{g}_{f,1},\ldots,\mathbf{0}] \tag{28}$$

with $\mathbf{0}$ being the $M_1 \times 1$ zeros vector.

Furthermore, the FB smoothing version of $\hat{\mathbf{R}}_{f,r}$ is

$$\hat{\mathbf{R}}_{f,r}^{\mathrm{FB}} = \frac{1}{2}\left(\hat{\mathbf{R}}_{f,r} + \mathbf{J}_{M_r}\hat{\mathbf{R}}_{f,r}^{*}\mathbf{J}_{M_r}\right) \tag{29}$$

where $\mathbf{J}_{M_r} \in \mathbb{C}^{M_r\times M_r}$ is the exchange matrix.

The expected value of $\hat{\mathbf{R}}_{f,r}^{\mathrm{FB}}$ is

$$\mathbf{R}_{f,r}^{\mathrm{FB}} = \frac{1}{2}\left(\mathbf{R}_{f,r} + \mathbf{J}_{M_r}\mathbf{R}_{f,r}^{*}\mathbf{J}_{M_r}\right) = \mathbf{G}_r\tilde{\mathbf{B}}_{f,r}\mathbf{G}_r^H + \sigma_q^2\mathbf{I}_{M_r} \tag{30}$$

where

$$\tilde{\mathbf{B}}_{f,r} = \frac{1}{2}\left(\mathbf{B}_{f,r} + \mathbf{D}_r\mathbf{B}_{f,r}^{*}\mathbf{D}_r\right) \tag{31}$$
$$\mathbf{D}_r = \mathrm{diag}\left(e^{jM_r\omega_{1,r}}, e^{jM_r\omega_{2,r}},\ldots,e^{jM_r\omega_{F,r}}\right). \tag{32}$$

Taking EVD on $\hat{\mathbf{R}}_{f,r}^{\mathrm{FB}}$ yields

$$\hat{\mathbf{R}}_{f,r}^{\mathrm{FB}} = \hat{\mathbf{V}}_{f,rs}\hat{\mathbf{\Lambda}}_{f,rs}\hat{\mathbf{V}}_{f,rs}^H + \hat{\mathbf{V}}_{f,rn}\hat{\mathbf{\Lambda}}_{f,rn}\hat{\mathbf{V}}_{f,rn}^H, \quad r = 2,3,\ldots,R,\ f = 1,2,\ldots,F \tag{33}$$

where the column vectors of $\hat{\mathbf{V}}_{f,rs} \in \mathbb{C}^{M_r\times 1}$ and $\hat{\mathbf{V}}_{f,rn} \in \mathbb{C}^{M_r\times(M_r-1)}$ are the eigenvectors that span the signal and noise subspaces of $\hat{\mathbf{R}}_{f,r}$, respectively, with the associated eigenvalues being the diagonal elements of $\hat{\mathbf{\Lambda}}_{f,rs}$ and $\hat{\mathbf{\Lambda}}_{f,rn}$.

Define $\hat{\mathbf{E}}_{f,r}$ of the form:

$$\hat{\mathbf{E}}_{f,r} = \mathbf{I}_{M_r} - \hat{\mathbf{V}}_{f,rs}\hat{\mathbf{V}}_{f,rs}^H, \quad r = 2,3,\ldots,R,\ f = 1,2,\ldots,F. \tag{34}$$

We see that the root-MUSIC method can also be used to obtain the fth component in the rth dimension frequency, $\hat{\omega}_{f,r}$, which are automatically paired with $\hat{\omega}_{f,1}$ with the use of $\hat{\mathbf{P}}_f^{\perp}$. The corresponding null-spectrum function is

$$f_{f,r}(z) = \beta_{f,r}^T(z^{-1})\,\hat{\mathbf{E}}_{f,r}\,\beta_{f,r}(z) \tag{35}$$

where $\beta_{f,r}(z) = [z, z^2,\ldots,z^{M_r}]^T$. The polynomial $f_{f,r}(z)$ has $2(M_r-1)$ roots and the rth dimension frequency of fth component $\hat{\omega}_{f,r}$ is obtained from the largest-magnitude root inside the unit circle according to $z = e^{-j\omega}$, $r = 2,3,\ldots,R$, $f = 1,2,\ldots,F$.

To summarize, the steps in the overall estimation procedure are:

1) Compute $\hat{\mathbf{R}}_1^{\mathrm{FB}}$ in (14) and perform its EVD
2) Compute $\hat{\mathbf{E}}_1$ using (19) and use (20) to compute $\hat{\omega}_{f,1}, f = 1,2,\ldots,F$
3) Construct $\hat{\mathbf{P}}_f^{\perp}$ in (21) and compute $\hat{\mathbf{R}}_{f,r}^{\mathrm{FB}}$ using (29)
4) Take the EVD on $\hat{\mathbf{R}}_{f,r}^{\mathrm{FB}}$, compute $\hat{\mathbf{E}}_{f,r}$ according to (34) and use (35) to obtain $\hat{\omega}_{f,r}$, $r = 2,3,\ldots,R$, $f = 1,2,\ldots,F$

TABLE I
COMPUTATIONAL COMPLEXITY OF FB-ROOT-MUSIC ALGORITHM

Operation	Dimension	Multiplication
Construct $\hat{\mathbf{R}}_1^{FB}$	$M_1 \times M_1$	$M_1 M + 2M_1^3$
EVD of $\hat{\mathbf{R}}_1^{FB}$	$M_1 \times M_1$	$25M_1^3$
Compute $\hat{\mathbf{E}}_1$	$M_1 \times M_1$	$M_1^2 F$
Construct $\hat{\mathbf{R}}_{f,r}^{FB}$	$M_r \times M_r$	$FM(M_1+M_r) + 2M_r^3 F$
EVD of $\hat{\mathbf{R}}_{f,r}^{FB}$	$M_r \times M_r$	$25M_r^3 F$
Compute $\hat{\mathbf{E}}_{f,r}$	$M_r \times M_r$	$M_r^2 F$
Total		$\mathcal{O}\Big(M\big(M_1 + F\sum_{r=2}^{R}(M_1+M_r)\big)\Big)$

III. ALGORITHM ANALYSIS

A. Computational Complexity

The computational complexity of the proposed root-MUSIC algorithm is studied and the results are provided in Table I. In summary, its complexity is $\mathcal{O}(M(M_1 + F\sum_{r=2}^{R}(M_1+M_r)))$. On the other hand, the computational requirement in the IMDF method [6] is about $\mathcal{O}(M^3)$ and that of the HOSVD algorithm [10] is at least $\mathcal{O}(\prod_{r=1}^{R}(M_r - L_r + 1)L_r^3)$ where L_r is the number of spatial smoothing subarrays in the rth dimension, $r = 1,2,\ldots,R$. Clearly, the proposed method is more computationally attractive than [6] and [10].

B. Mean Square Error

Analogous to (18) and (33), we define

$$\mathbf{R}_1^{\mathrm{FB}} = \mathbf{V}_{1s}\mathbf{\Lambda}_{1s}\mathbf{V}_{1s}^H + \sigma_q^2\,\mathbf{V}_{1n}\mathbf{V}_{1n}^H \tag{36}$$
$$\mathbf{R}_{f,r}^{\mathrm{FB}} = \lambda_{f,rs}\mathbf{v}_{f,rs}\mathbf{v}_{f,rs}^H + \sigma_q^2\,\mathbf{V}_{f,rn}\mathbf{V}_{f,rn}^H \tag{37}$$

where $\mathbf{V}_{1s} = [\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_F]$, $\mathbf{V}_{1n} = [\mathbf{v}_{F+1},\mathbf{v}_{F+2},\ldots,\mathbf{v}_{M_1}]$, $\mathbf{\Lambda}_{1s} = [\lambda_1,\lambda_2,\ldots,\lambda_F]$ and $\mathbf{V}_{f,rn} = [\mathbf{v}_{f,r}^{(2)},\mathbf{v}_{f,r}^{(3)},\ldots,\mathbf{v}_{f,r}^{(M_r)}]$. Note that $\mathbf{v}_f$, $f = 1,2,\ldots,F$, are the signal subspace eigenvectors of the first dimension, λ_f are the associated eigenvalues with $\mathbf{v}_f$ while $\mathbf{V}_{1n}$ is the noise subspace of the first dimension. Meanwhile, $\mathbf{v}_{f,rs}$ and $\lambda_{f,rs}$ are the eigenvector and eigenvalue of the signal subspace of the fth component in the rth dimension frequency, respectively, and $\mathbf{V}_{f,rn}$ is the noise subspace corresponding to $\mathbf{v}_{f,rs}$.

Following [15], the MSEs of the first dimension frequency estimates are computed as

$$\mathbb{E}\{(\hat{\omega}_{f,1}-\omega_{f,1})^2\} = \frac{M_1 M_2}{2M}\,\frac{\sigma_q^2\sum_{f=1}^{F}\frac{\lambda_f}{\left(\lambda_f-\sigma_q^2\right)^2}|\mathbf{v}_f^H\mathbf{g}_{f,1}|^2}{\mathbf{d}_{f,1}^H\mathbf{V}_{1n}\mathbf{V}_{1n}^H\mathbf{d}_{f,1}} \tag{38}$$

where $\mathbf{d}_{f,1} = \mathrm{d}\,\mathbf{g}_{f,1}/\mathrm{d}\omega_{f,1}$, and $\mathbb{E}$ is the expectation operator.

On the other hand, the MSEs of the frequency estimates in the remaining dimensions are

$$\mathbb{E}\left\{(\hat{\omega}_{f,r}-\omega_{f,r})^2\right\} = \frac{M_1 M_r}{2M}\,\frac{\sigma_q^2\frac{\lambda_{f,rs}}{\left(\lambda_{f,rs}-\sigma_q^2\right)^2}\left|\mathbf{v}_{f,rs}^H\mathbf{g}_{f,r}\right|^2}{\mathbf{d}_{f,r}^H\mathbf{V}_{f,rn}\mathbf{V}_{f,rn}^H\mathbf{d}_{f,r}} \tag{39}$$

where $\mathbf{d}_{f,r} = \mathrm{d}\,\mathbf{g}_{f,r}/\mathrm{d}\omega_{f,r}$.

IV. SIMULATION RESULTS

Computer simulations have been conducted to evaluate the frequency estimation performance of the proposed approach for multiple 3-D and 4-D sinusoids in the presence of white Gaussian noise. The algorithms without and with FB smoothing, denoted by root

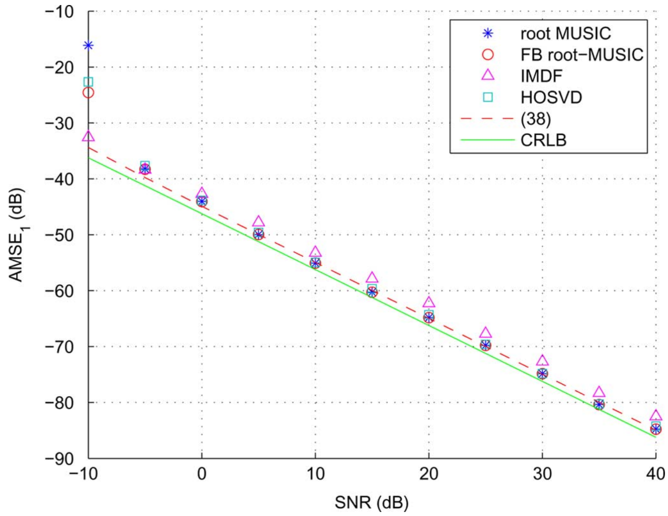

Fig. 1. AMSE_1 versus SNR with distinct frequencies.

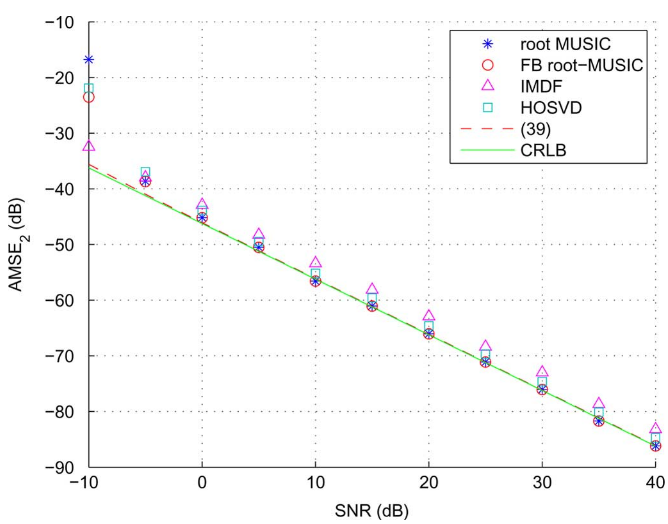

Fig. 2. AMSE_2 versus SNR with distinct frequencies.

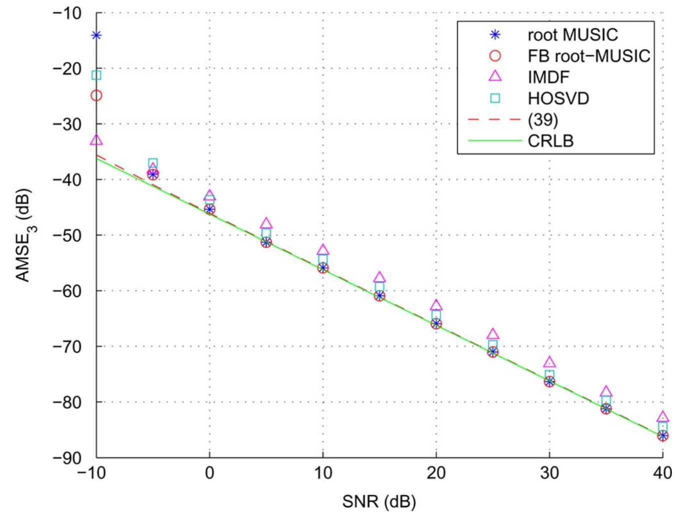

Fig. 3. AMSE_3 versus SNR with distinct frequencies.

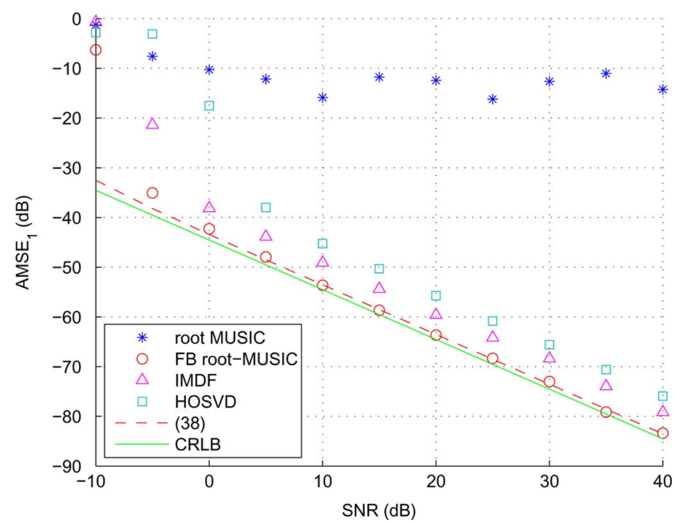

Fig. 4. AMSE_1 versus SNR with identical frequencies.

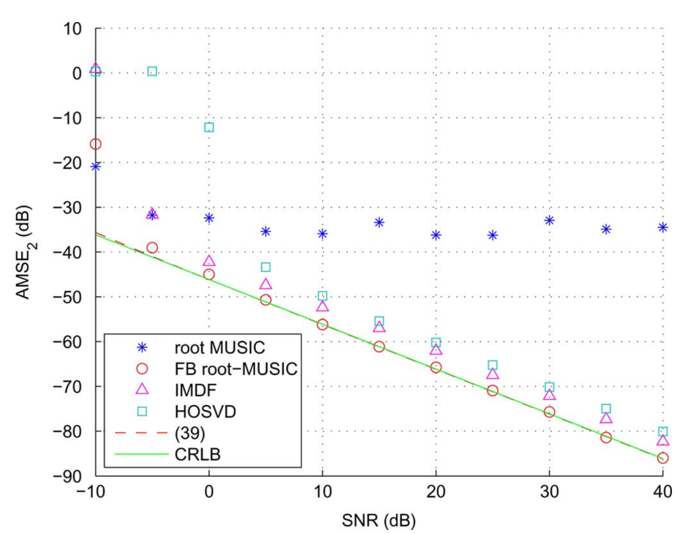

Fig. 5. AMSE_2 versus SNR with identical frequencies.

MUSIC and FB root MUSIC, are examined. The average MSE of the rth dimension, denoted by AMSE_r, is employed as the performance measure. All results provided are averaged from 200 independent runs. Apart from CRLB [6], [17] and [18], the performance of the proposed approach is compared with that of the IMDF [6], HOSVD [10] algorithms. The signal power is defined as $\sigma_s^2 = (\sum_{m_1=1}^{M_1} \cdots \sum_{m_R=1}^{M_R} |s_{m_1,m_2,\ldots,m_R}|^2)/M$ and $q_{m_1,m_2,\ldots,m_R}$ is scaled to produce different signal-to-noise ratio (SNR) which is defined as $\mathrm{SNR} = \sigma_s^2/\sigma_q^2$.

In the first test, we consider 3-D frequency estimation with $M_1 = M_2 = M_3 = 15$ and the number of tones is $F = 3$. The sinusoidal parameters are $\alpha_1 = 1$, $\{\omega_{1,r}\} = \{0.1\pi, 0.25\pi, 0.4\pi\}$, $\alpha_2 = e^{j0.3\pi}$, $\{\omega_{2,r}\} = \{0.25\pi, 0.4\pi, 0.1\pi\}$, $\alpha_3 = e^{j0.5\pi}$ and $\{\omega_{3,r}\} = \{0.4\pi, 0.1\pi, 0.25\pi\}$. The results of AMSE_r versus SNR are plotted in Figs. 1–3. It is observed that although the MSE of the proposed approach in the first dimension is comparable with that of HOSVD scheme, the former is superior to [6] and [10], with performance close to CRLB, in the remaining dimensions. The theoretical calculations of (38) and (39) also agree very well with the simulation results for sufficiently high SNR conditions. The above test is repeated with identical frequency scenario of $\{\omega_{1,r}\} = \{0.1\pi, 0.3\pi, 0.4\pi\}$, $\{\omega_{2,r}\} = \{0.25\pi, 0.3\pi, 0.4\pi\}$ and $\{\omega_{3,r}\} = \{0.4\pi, 0.35\pi, 0.45\pi\}$, and the results are plotted in Figs. 4–6. In this challenging case, the proposed

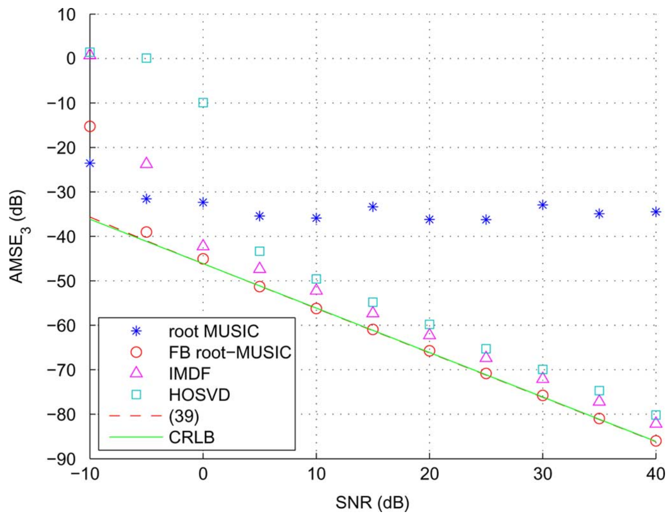

Fig. 6. AMSE_3 versus SNR with identical frequencies.

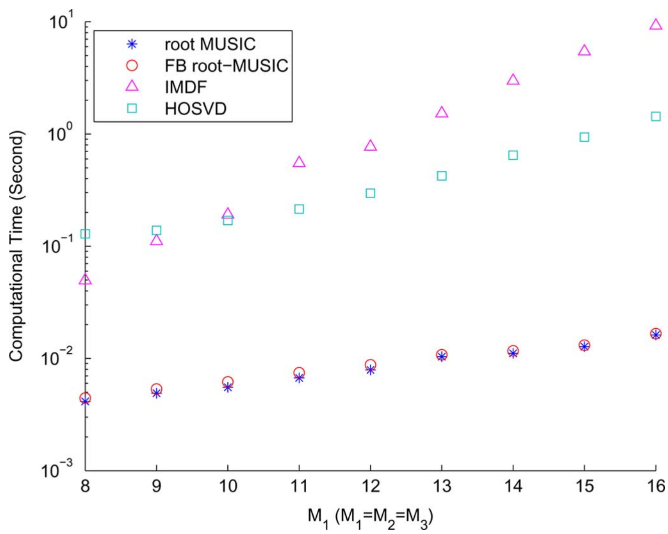

Fig. 7. Computational time versus M_1.

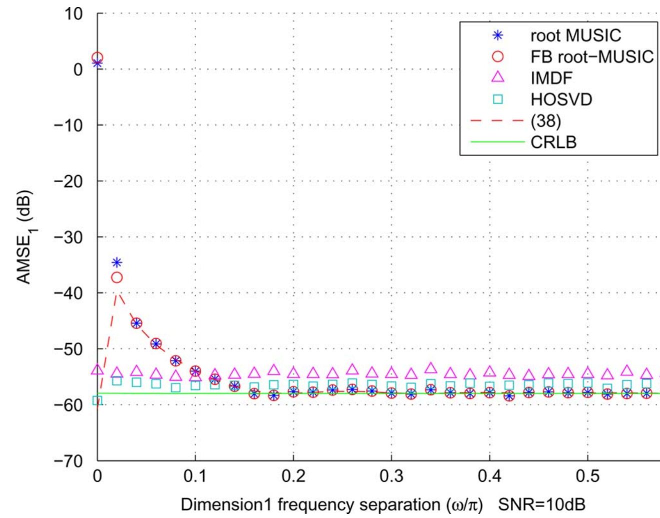

Fig. 8. AMSE_1 versus frequency separation.

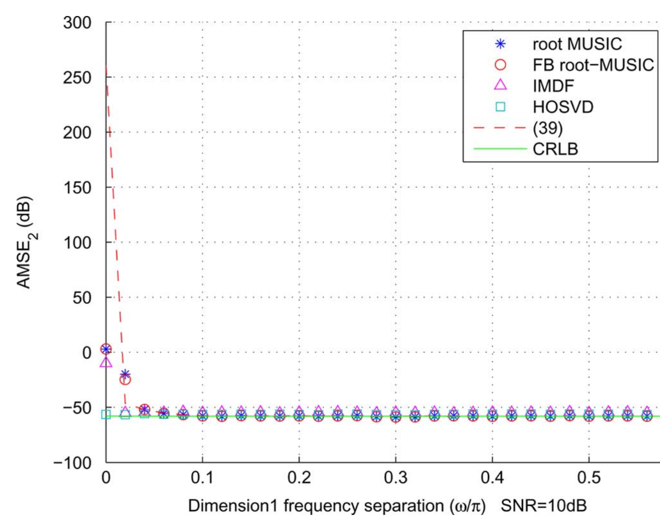

Fig. 9. AMSE_2 versus frequency separation.

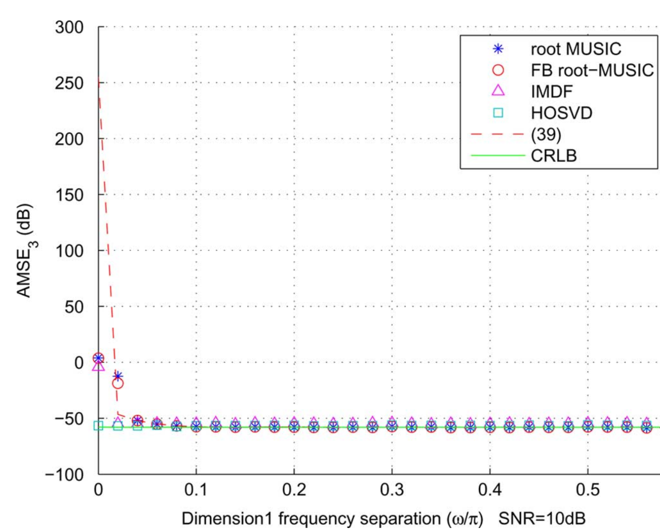

Fig. 10. AMSE_3 versus frequency separation.

method without FB smoothing performs the worst because there are identical frequencies in two dimensions. Nevertheless, the one with FB smoothing outperforms [6] and [10] for all dimensions and its performance attains CRLB in the second and third dimensions.

In the second experiment, the average computational time of the investigated algorithms versus different M_1 with $M_1 = M_2 = M_3$ in 3-D frequency estimation are plotted in Fig. 7. It is seen that the proposed approach is much more computationally efficient than the IMDF and HOSVD algorithms, which aligns with the complexity analysis in Section III.

In the third experiment, AMSE_r for different frequency separation by varying $\omega_{2,1}$ is studied. We consider two 3-D tones with $M_1 = M_2 = M_3 = 15$. The signal parameters are $\alpha_1 = 1$, $\alpha_2 = e^{j0.3\pi}$, $\{\omega_{1,r}\} = \{0.1\pi, 0.25\pi, 0.4\pi\}$, $\omega_{2,2} = 0.4\pi$ and $\omega_{2,3} = 0.1\pi$ while $\omega_{2,1}$ is varying from 0.1π to 0.6π. The corresponding results at $\mathrm{SNR} = 10$ dB are shown in Figs. 8–10. It is observed that the estimation performance of proposed approach is inferior to the IMDF and HOSVD schemes in the first dimension when the frequency separation is less than 0.1π. Nevertheless, its performance is comparable to them in the other two dimensions.

Finally, 4-D frequency estimation AMSE_r versus SNR is studied and the results are plotted in Figs. 11–14. The number of sinusoids is

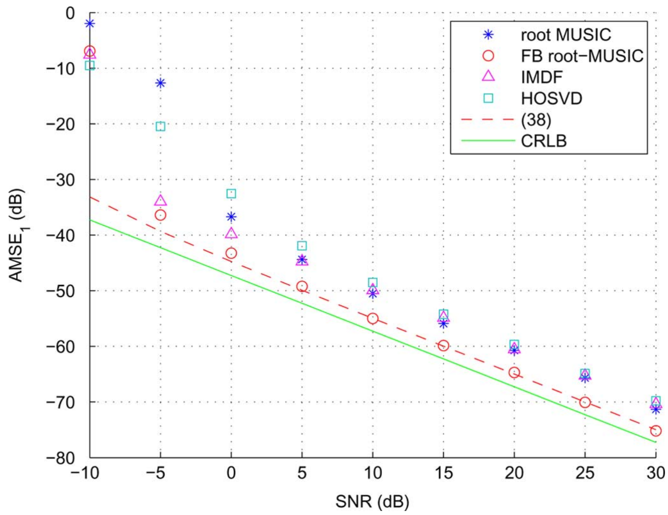

Fig. 11. AMSE_1 versus SNR for 4D Data.

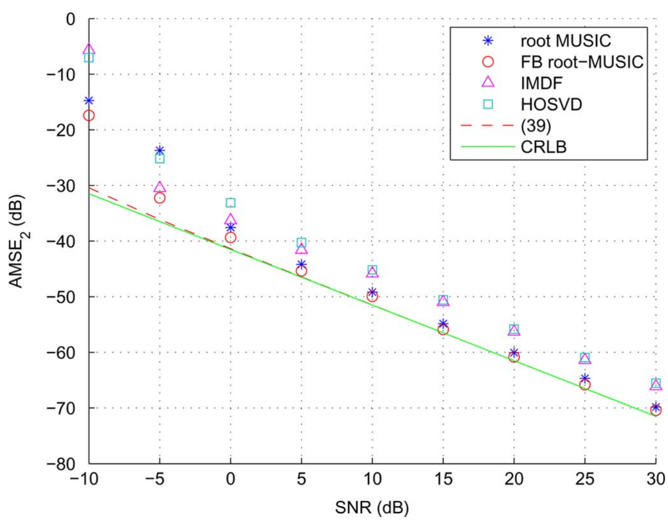

Fig. 12. AMSE_2 versus SNR for 4D Data.

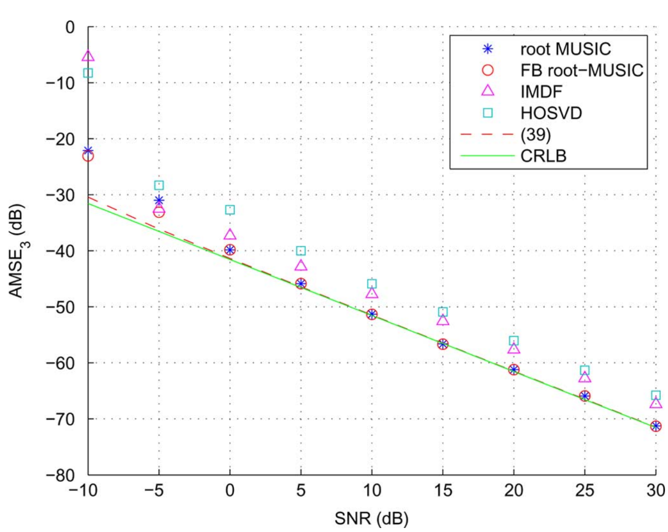

Fig. 13. AMSE_3 versus SNR for 4D Data.

$F = 3$ while $M_1 = 15, M_2 = M_3 = M_4 = 7$. The signal parameters are $\alpha_1 = 1$, $\{\omega_{1,r}\} = \{0.1\pi, 0.25\pi, 0.4\pi, 0.1\pi\}$, $\alpha_2 = e^{j0.3\pi}$,

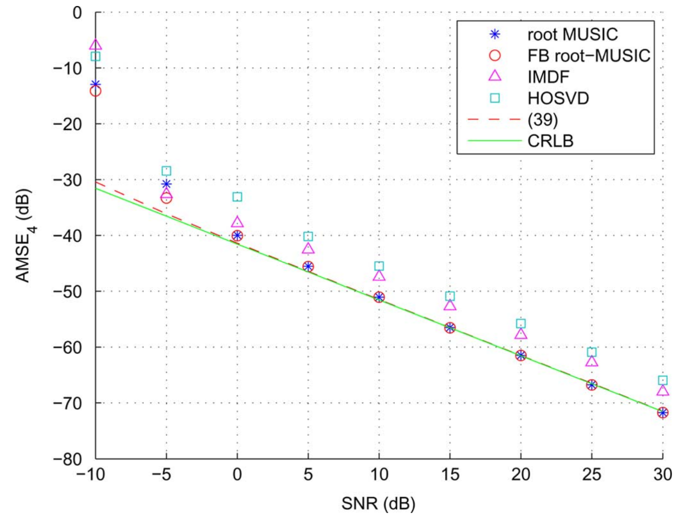

Fig. 14. AMSE_4 versus SNR for 4D Data.

$\{\omega_{2,r}\} = \{0.25\pi, 0.4\pi, 0.4\pi, 0.1\pi\}$, $\alpha_3 = e^{j0.5\pi}$ and $\{\omega_{3,r}\} = \{0.4\pi, 0.25\pi, 0.45\pi, 0.15\pi\}$.

Again, we observe that the performance of the proposed estimation is comparable to those of the other two methods [6] and [10]. However, the proposed method attains the CRLB and is better than other the two methods for dimensions 2, 3, 4 at sufficiently high SNRs.

V. Conclusion

A new approach for frequency estimation of multidimensional sinusoids in additive white circular Gaussian noise has been developed. The main idea in our methodology is to rearrange the R-D sinusoids as a series of 2-D slice matrices and combine the subspace and projection separation techniques. The frequencies in one dimension are first estimated using the root-MUSIC algorithm, which are then utilized to separate the frequencies in the remaining dimensions. Using the separated data, the remaining frequencies are then estimated one by one using the root-MUSIC method such that the multidimensional parameters are automatically paired. It is shown that the proposed algorithm is superior to the IMDF and HOSVD methods in terms of computational load and estimation performance.

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Computing Constrained Cramér-Rao Bounds

Paul Tune

***Abstract*—We revisit the problem of computing submatrices of the Cramér-Rao bound (CRB), which lower bounds the variance of any unbiased estimator of a vector parameter $\boldsymbol{\theta}$. We explore iterative methods that avoid direct inversion of the Fisher information matrix, which can be computationally expensive when the dimension of $\boldsymbol{\theta}$ is large. The computation of the bound is related to the quadratic matrix program, where there are highly efficient methods for solving it. We present several methods, and show that algorithms in prior work are special instances of existing optimization algorithms. Some of these methods converge to the bound monotonically, but in particular, algorithms converging nonmonotonically are much faster. We then extend the work to encompass the computation of the CRB when the Fisher information matrix is singular and when the parameter $\boldsymbol{\theta}$ is subject to constraints. As an application, we consider the design of a data streaming algorithm for network measurement.**

***Index Terms*—Cramér-Rao bound, Fisher information, matrix functions, optimization, quadratic matrix program.**

I. Introduction

The Cramér-Rao bound (CRB) [11] is important in quantifying the best achievable covariance bound on unbiased parameter estimation of n parameters $\boldsymbol{\theta}$. Under mild regularity conditions, the CRB is asymptotically achievable by the maximum likelihood estimator. The computation of the CRB is motivated by its importance in various engineering disciplines: medical imaging [6], blind system identification [18], and many others.

A related quantity is the Fisher information matrix (FIM), whose inverse is the CRB. Unfortunately, direct inversion techniques are known for their high complexity in space ($O(n^2)$ bytes of storage) and time ($O(n^3)$ floating point operations or flops). Often, one is just interested in a portion of the covariance matrix. In medical imaging applications, for example, only a small region is of importance, which is related to the location of a tumor or lesion. In this instance, computing the full inverse of the FIM becomes especially tedious and intractable when the number of parameters is large. In some applications, the FIM itself is singular, and the resulting Moore-Penrose pseudoinverse computation is even more computationally demanding. Avoiding the additional overhead incurred from direct inversion or other forms of matrix decompositions (Cholesky, QR, LU decompositions, for example) becomes a strong motivation.

Prior work [7], [8] proves the tremendous savings in memory and computation by presenting several recursive algorithms computing only submatrices of the CRB. Hero and Fessler [7] developed algorithms based on matrix splitting techniques, and statistical insight from the Expectation-Maximization (EM) algorithm. Only $O(n^2)$ flops are required per iteration, which is advantageous if convergence is rapid, and the algorithms produce successive approximations that converge monotonically to the CRB. Exponential convergence was

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The author is with the School of Mathematical Sciences, The University of Adelaide, Australia (e-mail: paul.tune@adelaide.edu.au).

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